

# Some Applications of $pq$ -groups in Graph Theory

Geoffrey Exoo  
Department of Mathematics and Computer Science  
Indiana State University  
Terre Haute, IN 47809  
g-exoo@indstate.edu

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## Abstract

We describe some new applications of nonabelian  $pq$ -groups to construction problems in Graph Theory. The constructions include the smallest known trivalent graph of girth 17, the smallest known regular graphs of girth five for several degrees, along with four edge colorings of complete graphs that improve lower bounds on classical Ramsey numbers.

For primes  $p$  and  $q$  such that  $p < q$  and  $q \equiv 1 \pmod{p}$ , there exists a nonabelian group of order  $pq$ , which is a semidirect product of  $Z_q$  by  $Z_p$  [3]. In this note, we use these groups to construct several graphs and graph colorings.

Let  $\mathcal{G}$  be a nonabelian  $pq$ -group and let  $a$  and  $b$  be elements of orders  $p$  and  $q$ , respectively. The subgroup generated by  $b$  is normal in  $\mathcal{G}$  and  $a^{-1}ba = b^r$  where  $r^p \equiv 1 \pmod{q}$ . So the elements of  $\mathcal{G}$  can be represented in the form  $a^i b^j$ , for  $0 \leq i < p$  and  $0 \leq j < q$ . Group multiplication can be defined by

$$a^i b^j * a^s b^t = a^{i+s} b^{jr^s+t}.$$

In what follows, it will be convenient to refer to elements of a  $pq$ -group using the notation  $(i, j)$  instead of  $a^i b^j$ .

## 1 Ramsey Graphs

Recall that an  $(s, t)$ -coloring of the complete graph  $K_n$  is a 2-coloring of its edges such that there is neither a complete subgraph of order  $s$ , all of whose edges are color 1, nor a complete subgraph of order  $t$ , all of whose edges are color 2. The Ramsey number  $R(s, t)$  is the minimum  $n$  such that no  $(s, t)$  coloring on  $K_n$  exists.

Our first application of  $pq$ -groups gives us a new lower bound for  $R(4, 8)$ . For this we use a Cayley coloring on the nonabelian group  $\mathcal{G}$  of order 55, where we take  $p = 5$ ,  $q = 11$ , and  $r = 3$ . Note that in a Cayley coloring, we assign

a color to each element  $g \in \mathcal{G}$ , and then assign that color to edges of the form  $(x, xg)$ , for all  $x \in \mathcal{G}$ . In the array below the entry in row  $i$ , column  $j$  gives the color assigned to each element  $(i, j)$  i.e.,  $a^i b^j$ , of  $\mathcal{G}$ . It can be verified that this indeed determines a  $(4, 8)$ -coloring of  $K_{55}$ , thereby improving the lower bound on  $R(4, 8)$  to 56 [4].

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1 2 2 1 1 1 1 1 2 2
2 2 2 1 1 2 1 1 2 2
2 1 2 2 2 2 1 2 2 1
2 2 2 2 1 2 2 1 2 1
2 1 2 1 2 2 1 2 1 2

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To obtain more new Ramsey colorings we use a nice property of  $pq$ -groups: they have a natural permutation representation of degree  $q$ . This is applied to the construction of Ramsey graphs on  $q$  vertices by considering the induced action on the edges of  $K_q$ . This way we only need to make  $\frac{q-1}{2p}$  color choices, one for each orbit, and this makes dealing with searches for relatively large Ramsey graphs much easier.

Using this permutation representation, along with the combinatorial search technique outlined in [1], we able to improve three classical Ramsey bounds [4]. These bound may be of particular interest, since they are not circle colorings.

$$R(3, 27) \geq 157$$

$$R(3, 31) \geq 197$$

$$R(5, 17) \geq 283$$

Since these graphs are so large, rather than attempt to present them here, we note that they can be obtained in several electronic formats from the author's web site at <http://ginger.indstate.edu/ge/RAMSEY>.

## 2 A Trivalent Graph of Girth 17

In this section we use a construction method, first discussed in [2], that can be viewed as a generalization of Cayley graphs. Instead of using one copy of a group as the vertex set, as with Cayley graphs, we use multiple copies. The groups that seem to be most useful for the cage problems mentioned below are the nonabelian  $pq$ -groups (recall that a  $(k, g)$ -cage is a smallest  $k$ -regular graph of girth  $g$ ). A property of  $pq$ -groups that makes them useful in this regard is the absence of elements of order less the  $p$  (save the identify).

Our next construction is made by using the nonabelian group  $\mathcal{G}$  of order  $301 = 7 \times 43$ . To complete the definition of the group, we take  $r = 4$ , then start with a set  $V$  of 2408 vertices, viewed as 8 copies of  $\mathcal{G}$ . Define a group action on  $V$  by fixing an ordering of the group elements,  $g_0, \dots, g_{300}$ , and identify the

vertices with the integers  $0 \dots 2407$ . The ordering of the group elements is given by having  $(i_1, j_1) < (i_2, j_2)$  if and only if  $i_1 < i_2$  or  $i_1 = i_2$  and  $j_1 - 1 < j_2$ . We associate vertex  $301i + j$  with group element  $g_j$  for  $1 \leq i \leq 8$ . Then the action of  $\mathcal{G}$  of  $V$  is defined as follows. For  $g \in \mathcal{G}$ , let  $g(301i + j) = 301i + k$  if and only if  $g_j g = g_k$  in  $\mathcal{G}$ . Then it is simply a matter of selecting a set of orbits from the induced action on the edges. The orbits used are indicated below, where we list one edge from each edge orbit chosen.

0	593
0	1185
0	1329
301	1343
301	2206
602	1158
602	1967
602	2278
903	1506
1204	2006
1505	1923
1505	2374

Adjacency lists for all the cage constructions presented here can be obtained from <http://ginger.indstate.edu/ge/CAGES>.

Similarly, we were able to construct the smallest known four-regular graph of girth 9 [5] from five copies of the nonabelian group  $G$  of order 55, using  $p = 5$ ,  $q = 11$ ,  $r = 3$ . Edge orbit representatives are again listed below.

0	55
0	73
0	212
0	239
55	219
55	227
110	176
110	214
110	258
110	263

### 3 Small Regular Graphs of Girth 5

Finally, we undertook a search for Cayley graphs of girth 5 on nonabelian  $pq$ -groups. We were able to find the smallest known regular graphs of girth 5 for several even degrees in the range 12 to 32. The table below summarizes our results, listing the degree, girth, and order of the graph along with the values of  $p$ ,  $q$ , and  $r$  for the group. The column labeled "ratio" gives the order of the graph divided by the Moore bound.

degree	girth	order	p	q	r	ratio
12	5	203	7	29	7	1.400
14	5	355	5	71	5	1.802
16	5	497	7	71	20	1.934
18	5	655	5	131	53	2.015
20	5	889	7	127	2	2.217
22	5	1027	13	79	8	2.118
24	5	1255	5	251	20	2.175
26	5	1655	5	331	64	2.445
28	5	2005	5	401	39	2.554
30	5	2359	7	337	8	2.618
32	5	2947	7	421	33	2.875

Below we give the generators for each of the Cayley graphs. An entry of  $(i, j)$  means that the group element  $a^i b^j$  is a generator. The inverses of these generators are not listed.

For  $d = 12$ ,  $n = 203$ :

$(1,0)(2,2)(2,27)(2,3)(3,11)(3,28)$

For  $d = 14$ ,  $n = 355$ :

$(0,10)(0,61)(1,54)(1,69)(2,35)(2,41)(2,53)(2,64)$

For  $d = 16$ ,  $n = 497$ :

$(1,24)(1,41)(1,51)(2,20)(2,22)(2,45)(3,4)(3,56)$

For  $d = 18$ ,  $n = 655$ :

$(1,10)(1,113)(1,29)(1,37)(1,87)(2,118)(2,62)(2,86) (2,97)$

For  $d = 20$ ,  $n = 889$ :

$(1,112)(1,122)(1,21)(1,75)(1,90)(1,98)(2,50)(2,56)$   
 $(2,59)(3,41)$

For  $d = 22$ ,  $n = 1027$ :

$(1,22)(1,30)(1,52)(2,0)(2,27)(2,39)(3,27)(3,71)(4,13)$   
 $(5,14)(5,16)$

For  $d = 24$ ,  $n = 1255$ :

$(1,140)(1,159)(1,198)(1,246)(1,33)(1,63)(2,110)(2,126)$   
 $(2,150)(2,173)(2,209)(2,32)$

For  $d = 26$ ,  $n = 1655$ :

(1,108)(1,109)(1,260)(1,265)(1,267)(1,38)(2,107)(2,191)  
(2,215)(2,237)(2,25)(2,312)(2,54)

For  $d = 28$ ,  $n = 2005$ :

(1,109)(1,140)(1,166)(1,376)(1,390)(1,92)(2,133)(2,161)  
(2,196)(2,204)(2,340)(2,372)(2,377)(2,40)

For  $d = 30$ ,  $n = 2359$ :

(1,183)(1,201)(1,302)(1,334)(1,75)(2,140)(2,181)(2,233)  
(2,293)(2,306)(3,179)(3,238)(3,327)(3,4)(3,96)

For  $d = 32$ ,  $n = 2947$ :

(1,190)(1,214)(1,328)(1,336)(1,83)(2,161)(2,220)(2,340)  
(2,341)(2,360)(3,25)(3,253)(3,330)(3,334)(3,340)(3,392)

## References

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